* Where does partial fraction decomposition come from?

↳ Consider adding $\frac{3}{x+4} + \frac{2}{x-3}$. The result is $\frac{5x-1}{x^2+x-12}$, to reverse the procedure, start with $\frac{5x-1}{x^2+x-12}$ and write it as the sum (or diff.) of two simpler fractions. This process is partial fraction decomposition, and the two simpler fractions are called partial fractions.

$\boxed{\text{10.5 - Partial Fraction Decomposition}}$

CASE 1

Decompose $\frac{P}{Q}$, where Q has only non-repeated linear factors: $Q(x) = (x-a_1)(x-a_2) \cdot \ldots \cdot (x-a_n)$ where none of the numbers $a_1, a_2, \ldots, a_n$ is equal. In this case, partial fraction decomposition of $\frac{P}{Q}$ is of the form:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots + \frac{A_n}{x-a_n}$$

where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

Ex. 1 Write the partial fraction decomposition of $\frac{x}{x^2-5x+6}$

① Factor denominator $(x-3)(x-2)$ and conclude that denom. contains only non-repeated linear factors.

② Decompose the rational expression according to case 1

$$\frac{x}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

③ To find A & B, clear fractions by multiplying each side by $(x-2)(x-3)$:

$$(x-2)(x-3)\left[\frac{x}{x^2-5x+6}\right] = \left[\frac{A}{x-2} + \frac{B}{x-3}\right](x-2)(x-3)$$

↳ $x = A(x-3) + B(x-2) \rightarrow x = Ax - 3A + Bx - 2B$

↳ $x = x(A+B) + (-3A-2B)$

④ Set up 2 equ./2 unknowns and solve for A & B

$1 = A + B$ equate coefficients of x: $1x = (A+B)x$

$0 = -3A - 2B$ equate coefficients of x^0: $0x^0 = (-3A-2B)x^0$

↳ $A = -2$, $B = 3$

⑤ Write as partial fraction decomposition: $\boxed{\frac{x}{x^2-5x+6} = \frac{-2}{x-2} + \frac{3}{x-3}}$

For ex. 1, you could also solve by letting $x=2$ and the term containing B drops out, leaving $2=A(-1)$ or $\boxed{A=-2}$

$2=A[(2)-3]+B[(2)-2] \rightarrow 2=A(-1) \rightarrow \underline{-2=A}$

$3=A[(3)-3]+B[(3)-2] \rightarrow \underline{3=B}$

- Then let $x=3$, so the term containing A drops out, leaving $\boxed{B=3}$

Case 2: Decompose $\frac{P}{Q}$ where Q has repeated linear factors

If the polynomial Q has a repeated linear factor, say $(x-a)^n$, where $n \geq 2$ and is an integer, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for

$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\dots+\frac{A_n}{(x-a)^n}$ where $A_1, A_2, \dots, A_n$ are to be determined

Ex. 2 Write the partial fraction decomposition of $\frac{x+2}{x^3-2x^2+x}$

(1) Factor the denominator

$x(x^2-2x+1)=x(x-1)(x-1)=x(x-1)^2$

- and see that the denom has a non-repeated linear factor x and the repeated linear factor $(x-1)^2$.

- By case 1, we allow for $\frac{A}{x}$ in the decomposition and, by case 2, we allow for $\frac{B}{x-1}+\frac{C}{(x-1)^2}$ to account for the repeated linear factor of $(x-1)^2$

(2) Write as $\frac{x+2}{x^3-2x^2+x}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ and then clear fractions by mult. each side by $x^3-2x^2+x=\underline{x(x-1)^2}$

$$x(x-1)^2\left[\frac{x+2}{x^3-2x^2+x}\right]=\left[\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\right]x(x-1)^2$$

to get $x+2=A(x-1)^2+B[x(x-1)]+C(x)$

(3) let $x=0$ to get B+C to drop out $\rightarrow 2=A(-1)^2$ or $\boxed{A=2}$

(4) let $x=1$ to get A+B to drop out $\rightarrow 3=C$ or $\boxed{C=3}$

- now you're left w/ $x+2=\underline{2}(x-1)^2+Bx(x-1)+\underline{3}x$

(5) let $x=2$ (any choice other than $x=0$ or 1 will work) and solve for B

$4=2(1)^2+B(2)(1)+3(2) \rightarrow \boxed{B=-2}$, so $\boxed{\frac{x+2}{x^3-2x^2+x}=\frac{2}{x}+\frac{-2}{x-1}+\frac{3}{(x-1)^2}}$